CSE 1200 Calculus — Complete Summary

Lecture 1: Limits, Continuity, L'Hopital's Rule

Key Concepts

- **Indeterminate form 0/0**: direct substitution gives $0/0$; the limit may still exist
- **L'Hopital's Rule**: if $\lim \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

  provided the right-hand limit exists. Can be applied repeatedly.
- **Continuity**: f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Method: Evaluate a 0/0 Limit

1. **Factor and cancel**: factor numerator and denominator, cancel common $(x-a)$ factors
2. **Conjugate multiplication**: for $\sqrt{\cdot}$ expressions, multiply by conjugate: $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$
3. **L'Hopital's Rule**: differentiate top and bottom separately (NOT the quotient rule)
4. **Substitution**: let $u = x - a$ to center the limit at 0

Lecture 2: Indeterminate Forms, Squeeze Theorem, Limits at Infinity

Key Concepts

- **Indeterminate powers** (1^∞, 0^0, ∞^0): arise with $[f(x)]^{g(x)}$
- **Squeeze Theorem**: if $g(x) \le f(x) \le h(x)$ and $\lim g = \lim h = L$, then $\lim f = L$
- Useful when the expression contains bounded oscillating terms like $\sin(x)$ or $e^{\sin(x)}$
- **Bounds to know**: $-1 \le \sin(x) \le 1$, $e^{-1} \le e^{\sin(x)} \le e$

Method: Evaluate $f(x)^{g(x)}$ (Exponential Rewrite)

1. Write $f(x)^{g(x)} = e^{g(x)\ln(f(x))}$
2. The limit becomes $e^{\lim g(x)\ln(f(x))}$
3. Rewrite $g(x)\ln(f(x))$ as $\frac{\ln(f(x))}{1/g(x)}$ to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$
4. Apply L'Hopital on the exponent
5. **Special pattern**: $\lim_{u\to 0}(1+u)^{1/u} = e$

Method: Limits at Infinity

1. **Divide by dominant term**: for rational-like functions, divide numerator and denominator by x^n
2. **Conjugate trick**: for $\sqrt{x^2 + ax} - x$, multiply by conjugate $\to \frac{ax}{\sqrt{x^2+ax}+x} \to \frac{a}{2}$
3. **Squeeze**: bound oscillating parts, show the bounds converge to the same limit

WARNING

- L'Hopital can fail: $\lim \frac{x}{x+\sin x}$ gives $\frac{1}{1+\cos x}$ which does not exist. Instead divide by x: $\frac{1}{1+\sin(x)/x} \to 1$

Lecture 3: Asymptotes, Inverse Functions, Inverse Trig

Key Concepts — Asymptotes

- **Vertical asymptote** $x = a$: $\lim_{x\to a} f(x) = \pm\infty$ (denominator zero, numerator nonzero)
- **Horizontal asymptote** $y = L$: $\lim_{x\to\pm\infty} f(x) = L$
- **Oblique asymptote** $y = ax + b$: when no horizontal asymptote exists
  - $a = \lim_{x\to\pm\infty} \frac{f(x)}{x}$, then $b = \lim_{x\to\pm\infty}(f(x) - ax)$

WARNING

- If the limit at a candidate vertical asymptote is **finite**, it is a **removable singularity**, not an asymptote
- Check $x \to +\infty$ and $x \to -\infty$ separately — they can give different asymptotes

Key Concepts — Inverse Functions

- **Swap and solve**: write $y = f(x)$, swap x and y, solve for y to get $f^{-1}(x)$
- For quadratics: use the quadratic formula after swapping; choose the correct $\pm$ sign based on the restricted domain
- **Domain of** f^{-1} = range of f

Key Concepts — Inverse Trig Simplification

- **Triangle method**: for $\tan(\arcsin(x))$, draw a right triangle where $\sin(\theta) = x$, read off $\tan(\theta)$
- **Double angle**: $\cos(2\arcsin(x)) = 1 - 2x^2$
- **Range restrictions**: $\arccos(\cos(\theta)) = \theta$ only if $\theta \in [0, \pi]$; $\arctan(\tan(\theta)) = \theta$ only if $\theta \in (-\pi/2, \pi/2)$

Lecture 4: Linearization / Linear Approximation

Key Concepts

- **Linearization** of f at $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

- **Multivariable**: $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
- **From power series**: if $g(x) = \sum c_n(x - a)^n$, then $L(x) = c_0 + c_1(x - a)$
- **Error estimation** (differentials): $|\Delta f| \approx |f'(a)| \cdot |\Delta x|$

Common Approximations Near $x = 0$

Function	Approximation
$\sin(x)$	$\approx x$
$\cos(x)$	≈ 1
e^x	$\approx 1 + x$
$\ln(1 + x)$	$\approx x$
$(1 + x)^n$	$\approx 1 + nx$

Lecture 5: Implicit Differentiation, Mean Value Theorem, Newton-Raphson

Key Concepts — Implicit Differentiation

- Given an implicit equation $F(x, y) = 0$, find $\frac{dy}{dx}$ by differentiating both sides w.r.t. x, treating y as $y(x)$
- Chain rule: $\frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$

Method: Implicit Differentiation

1. Differentiate both sides with respect to x (apply chain rule to y-terms)
2. Collect all $\frac{dy}{dx}$ terms on one side
3. Factor out $\frac{dy}{dx}$ and isolate

- **Horizontal tangent**: set numerator of $\frac{dy}{dx} = 0$ (denominator $\neq 0$)
- **Vertical tangent**: set denominator of $\frac{dy}{dx} = 0$ (numerator $\neq 0$)

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Geometrically: there is a point where the tangent line is parallel to the secant line - **Word problems**: average velocity = instantaneous velocity at some point

Method: Newton-Raphson

- Iterative root-finding formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- **Graphical**: draw tangent at $(x_n, f(x_n))$; where it crosses the x-axis = x_{n+1}
- To sketch: choose $f(x)$ such that root = desired value (e.g., for $\sqrt[3]{6}$: $f(x) = x^3-6$), pick x_0 nearby, draw successive tangent lines converging to the root

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Lecture 6: Extreme Values on Closed Interval (Single Variable)

Method: Find Absolute Max/Min on $[a, b]$

1. Find critical points: solve $f'(x) = 0$ and find where $f'(x)$ is undefined, within (a, b)
2. Evaluate f at all critical points and at the endpoints $x = a$ and $x = b$
3. Compare: largest value = absolute max, smallest = absolute min

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Lecture 7: Integration Techniques (Substitution, By Parts)

Key Concepts

- **U-substitution**: identify inner function $u = g(x)$ such that $g'(x)$ appears in the integrand; replace $dx = du/g'(x)$
- **Integration by parts**: $\int u\,dv = uv - \int v\,du$
- **LIATE rule** for choosing u: **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rigonometric, **E**xponential (pick u from the left, dv from the right)
- **Reduction formulas**: express I_n in terms of I_{n-1} by applying IBP once
- **Combined techniques**: sometimes substitute first to simplify, then apply IBP

Lecture 8: Applications of Integration (Area, Symmetry, FTC)

Symmetry in Definite Integrals

- **Odd function** ($f(-x) = -f(x)$): $\int_{-a}^{a} f(x)\,dx = 0$
- **Even function** ($f(-x) = f(x)$): $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
- **Products**: odd $\times$ odd $=$ even, odd $\times$ even $=$ odd, even $\times$ even $=$ even

Method: Area Between Curves

1. Find intersection points: set $f(x) = g(x)$ and solve for x
2. Determine which function is on top on each sub-interval

3. Integrate: $A = \int_a^b |f(x) - g(x)|\, dx$
4. If curves cross, split into sub-intervals and sum the areas

Fundamental Theorem of Calculus

- **FTC Part 1**: if $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$
- **Chain rule variant**: if $g(x) = \int_a^{h(x)} f(t)\, dt$, then $g'(x) = f(h(x)) \cdot h'(x)$
- Critical points of g: set $g'(x) = f(x) = 0$

Lecture 9: Improper Integrals

Key Concepts

- **Type 1** (infinite limits): $\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$
- **Type 2** (singularity at c): split at c and take limits from each side
- **p-test**:
  - $\int_1^\infty \frac{1}{x^p}\, dx$ converges $\iff p > 1$
  - $\int_0^1 \frac{1}{x^p}\, dx$ converges $\iff p < 1$
- **Comparison test**: if $|f(x)| \leq g(x)$ and $\int g$ converges, then $\int f$ converges absolutely

Method: Evaluate an Improper Integral

1. Identify the type (infinite limit or singularity)
2. Replace the problematic bound with a limit variable
3. Evaluate the proper integral, then take the limit
4. Common techniques: substitution, IBP, comparison

WARNING

- Check for singularities at **every** point where the integrand is undefined, not just the endpoints

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Lecture 10: Recursive Sequences (Monotonicity, Boundedness, Limit)

Key Concepts

- Sequence defined by $a_0 = c$ and $a_{n+1} = f(a_n)$
- **Monotone Convergence Theorem**: a bounded monotone sequence converges

Method: Prove Convergence and Find the Limit

1. **Prove monotonicity** (by induction): show $a_{n+1} \geq a_n$ (increasing) or $a_{n+1} \leq a_n$ (decreasing)
2. **Prove boundedness** (by induction): show $a_n \leq M$ (if increasing) or $a_n \geq m$ (if decreasing)
3. Apply the Monotone Convergence Theorem $\to$ the limit L exists
4. **Find the limit**: assume $\lim a_n = L$, then $L = f(L)$. Solve the fixed-point equation
5. **Discard** solutions outside the range of the sequence

Lecture 11: Series — Geometric, Telescoping, Convergence Basics

Key Formulas

- **Geometric series**: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$
- **Differentiation trick**: $\sum nx^{n-1} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$, so $\sum nr^n = \frac{r}{(1-r)^2}$
- **Telescoping**: decompose a_n via partial fractions; consecutive terms cancel, leaving boundary terms

Recognizing Known Series

Series	Sum
$\sum \frac{x^n}{n!}$	e^x
$\sum(-1)^n \frac{x^{2n+1}}{2n+1}$	$\arctan(x)$
$\sum(-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$\sin(x)$
$\sum(-1)^n \frac{x^{2n}}{(2n)!}$	$\cos(x)$
$\sum(-1)^{n+1} \frac{x^n}{n}$	$\ln(1+x)$

Key Reference Series

- p**-series**: $\sum \frac{1}{n^p}$ converges $\iff p > 1$
- **Geometric**: $\sum r^n$ converges $\iff |r| < 1$
- **Harmonic**: $\sum \frac{1}{n}$ diverges ($p = 1$)

Lecture 12: Series Convergence / Divergence Tests

Decision Tree

1. **Divergence Test FIRST**: if $\lim a_n \neq 0$, the series **diverges**. Done.
2. **Check absolute convergence** (test $\sum |a_n|$):

- **Ratio test**: $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ absolutely convergent. Best for factorials and exponentials.
- **Root test**: $\lim |a_n|^{1/n} < 1 \Rightarrow$ absolutely convergent. Best for n-th powers.
- **Comparison / Limit Comparison**: compare with known p-series or geometric series.
- **Integral test**: $\sum f(n)$ converges $\iff \int_1^\infty f(x)\,dx$ converges (when f is positive, continuous, decreasing).

3. **If $\sum |a_n|$ diverges**, check **conditional convergence**: use the **Alternating Series Test** (b_n decreasing and $\lim b_n = 0$).

Examples

- $\sum(-1)^n \frac{n}{\sqrt{n^2-3}}$: $\lim \frac{n}{\sqrt{n^2-3}} = 1 \neq 0$. **Diverges** by divergence test.
- $\sum(-1)^n n^2 e^{-n}$: ratio test on $|a_n| = n^2/e^n$ gives $\lim = 1/e < 1$. **Absolutely convergent**.
- $\sum \frac{1+\sin(n)}{n^2}$: since $0 \leq 1 + \sin(n) \leq 2$, compare with $\frac{2}{n^2}$ (convergent p-series). **Converges**.

Lecture 13: Power Series — Interval and Radius of Convergence

Key Concepts

- A **power series** centered at a: $\sum c_n (x-a)^n$
- **Radius of convergence** R: the series converges for $|x-a| < R$ and diverges for $|x-a| > R$
- At $x = a \pm R$ (boundary): must check separately

Method: Find Interval of Convergence

1. **Ratio test**: $L = \lim \left|\frac{c_{n+1}}{c_n}\right|$, then $R = \frac{1}{L}$. Works best for factorials, exponentials, products.
2. **Root test**: $L = \lim |c_n|^{1/n}$, then $R = \frac{1}{L}$. Best for $c_n = (\text{something})^n$.
3. **Non-standard forms**: if the series has x^{2n} or x^{3n+1}, substitute $u = x^2$ or $u = x^3$, find R for u, then convert back.
4. **Boundary check**: substitute $x = a + R$ and $x = a - R$ into the series, test each:
   - Alternating series test, p-series, comparison, limit comparison
5. **Write the interval**: e.g., $[a-R, a+R)$ if the left endpoint converges and the right diverges

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Lecture 14: Power Series Operations, Linearization from Series

Key Identity

If $g(x) = \sum c_n (x-a)^n$, then:

$$g^{(k)}(a) = k! \cdot c_k$$

Method: Compute $f^{(k)}(0)$ from a Power Series

1. Find the Taylor series of the inner function (e.g., arctan$(x) = \sum(-1)^n \frac{x^{2n+1}}{2n+1}$)
2. Multiply by the polynomial factor (e.g., $x^2 \cdot$ arctan$(x) = \sum(-1)^n \frac{x^{2n+3}}{2n+1}$)
3. Find the coefficient c_k of x^k
4. $f^{(k)}(0) = k! \cdot c_k$

Linearization from Power Series

- If $g(x) = \sum c_n (x-a)^n$, the linearization at $x = a$ is simply $L(x) = c_0 + c_1(x-a)$
- This is just reading off the constant and linear terms from the series

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Lecture 15: Taylor Series, Error Bounds, Limits via Taylor

Key Taylor/Maclaurin Expansions

Function	Expansion
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
sin(x)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
cos(x)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
ln$(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
$(1+x)^\alpha$	$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \dots$
arctan(x)	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Method: Taylor Series for Anti-Derivative + Error Bound

1. Start from a known series and substitute to get the series for $f(x)$
   - E.g., $f(x) = e^{x^2} \to \sum \frac{(x^2)^n}{n!} = \sum \frac{x^{2n}}{n!}$
2. **Integrate term by term**: $\int \sum c_n x^n \, dx = C + \sum \frac{c_n x^{n+1}}{n+1}$
3. For a definite integral: $\int_0^b f(x)\,dx = \sum \frac{c_n \cdot b^{n+1}}{n+1}$
4. **Error bound (alternating series)**: $|$error$| \leq |$first omitted term$|$
5. **Error bound (positive series)**: compare remainder with geometric tail or integral estimate

6. Sum terms until the error bound is met

Method: Evaluate a Limit via Taylor Expansion

1. Expand functions in the expression to sufficient order to cancel the indeterminate form
2. Substitute the expansions, simplify, cancel common factors
3. Evaluate the limit

Lecture 16: Multivariable Functions — Domains, Graphs, Contour Plots

Method: Sketch Maximal Domain

1. **Identify restrictions**: $\sqrt{\cdot}$ requires argument ≥ 0; $\ln(\cdot)$ requires argument > 0; denominators $\neq 0$
2. Solve each restriction as an inequality in x and y
3. The domain is the **intersection** of all regions
4. Sketch: dashed lines for strict inequalities, solid for non-strict

Method: Match Functions to Graphs / Contour Plots

1. **Check symmetry**: $f(x, y) = f(y, x)$ means symmetric about $y = x$; $f(-x, y) = -f(x, y)$ means odd in x
2. **Set one variable to zero**: examine $f(x, 0)$ and $f(0, y)$ to identify cross-sections
3. **Level curves** $f = c$: identify shapes (lines, circles, hyperbolas)
4. **Check special points**: evaluate at origin, along axes, along $y = x$
5. **Identify type**: quadratic forms (paraboloids, saddles), products (xy gives saddle), trig (periodic contours)

Lecture 17: Partial Derivatives, Gradient Vector

Key Concepts

- **Partial derivative** $f_x = \frac{\partial f}{\partial x}$: differentiate w.r.t. x, treat y as constant
- **Gradient**: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$
- ∇f points in the direction of **steepest ascent**
- $|\nabla f|$ gives the **maximum rate of change**
- $\nabla f = \mathbf{0}$ at critical points
- **Chain rule for composites**: $\frac{\partial}{\partial x}[e^{xy}] = ye^{xy}$

Method: Compute the Gradient

1. Compute $f_x = \frac{\partial f}{\partial x}$ (treat y as constant)
2. Compute $f_y = \frac{\partial f}{\partial y}$ (treat x as constant)

3. Evaluate at the given point (a, b)

Lecture 18: Directional Derivative

Key Concepts

- **Directional derivative**: $D_{\mathbf{u}}f = \nabla f \cdot \hat{\mathbf{u}}$ (rate of change of f in direction $\mathbf{u}$)
- **Maximum rate of change**: $\max D_{\mathbf{u}}f = |\nabla f|$, in direction $\frac{\nabla f}{|\nabla f|}$
- **Minimum rate of change**: $\min D_{\mathbf{u}}f = -|\nabla f|$, in direction $-\frac{\nabla f}{|\nabla f|}$ (steepest descent)

Method: Compute a Directional Derivative

1. Compute gradient: $\nabla f = (f_x, f_y)$
2. Evaluate at the given point
3. Normalize the direction: $\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$
4. Dot product: $D_{\mathbf{u}}f = \nabla f \cdot \hat{\mathbf{u}}$

Lecture 19: Critical Points, Second Derivative Test, Absolute Max/Min

Method: Find and Classify Critical Points (Multivariable)

1. Compute f_x and f_y
2. Solve $f_x = 0$ AND $f_y = 0$ simultaneously $\rightarrow$ critical points
3. Check which solutions lie inside the domain
4. Compute the **Hessian determinant** at each critical point:

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

5. Classify:

Condition	Classification
$D > 0$ and $f_{xx} > 0$	Local minimum
$D > 0$ and $f_{xx} < 0$	Local maximum
$D < 0$	Saddle point
$D = 0$	Inconclusive

When $D = 0$: use other arguments (e.g., show f values along different paths through the point differ).

Extreme Value Theorem

If f is continuous on a closed bounded domain D, then f attains an absolute max and min on D. These occur at interior critical points or on the boundary.

Method: Find Absolute Max/Min on a Closed Domain

1. **Interior**: find all critical points where $\nabla f = \mathbf{0}$ inside D; evaluate f at each
2. **Boundary**: parametrize each boundary segment (edge, curve) $\rightarrow$ reduce to single-variable optimization
   - For a triangle with vertices $(1,1)$, $(2,1)$, $(2,2)$: three edges
   - Edge: fix one variable or write $y = g(x)$, find critical points of $f(x, g(x))$
3. **Corners/vertices**: evaluate f at all corner points
4. **Compare**: largest value = absolute max, smallest = absolute min

Common domains: **rectangles** (4 edges), **triangles** (3 edges), **disks** (parametrize boundary with $x = r\cos\theta$, $y = r\sin\theta$)

Lecture 20: Complex Numbers — Arithmetic, Polar Form

Key Concepts

- **Complex number**: $z = a + bi$ where $i^2 = -1$
- **Modulus**: $|z| = \sqrt{a^2 + b^2}$
- **Conjugate**: $\bar{z} = a - bi$; note $z \cdot \bar{z} = |z|^2$
- **Polar form**: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$
- **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$

Operations

Operation	Rule
Division	$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$ (multiply by conjugate)
Powers	$z^n = r^n e^{in\theta}$
Modulus of product	$\|z_1 z_2\| = \|z_1\|\|z_2\|$
Argument of product	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
Conjugate properties	$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Lecture 21: Complex Roots, Euler's Formula

Method: Find All Roots of $z^n = c$

1. Convert c to polar: $c = re^{i\theta}$
2. Apply the root formula:

$$z_k = r^{1/n} \cdot e^{i(\theta + 2\pi k)/n}, \quad k = 0, 1, \dots, n-1$$

3. Convert to $a + bi$: $z_k = r^{1/n}(\cos\phi_k + i\sin\phi_k)$ where $\phi_k = \frac{\theta + 2\pi k}{n}$

4. The n roots are equally spaced on a circle of radius $r^{1/n}$, separated by angle $\frac{2\pi}{n}$
5. **For** $(z-a)^n = c$: solve $w^n = c$ first, then $z = w + a$

Proof of Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

1. Expand: $e^{i\theta} = \sum \frac{(i\theta)^n}{n!}$
2. Separate even and odd terms:
    - Even ($n = 2k$): $\sum(-1)^k \frac{\theta^{2k}}{(2k)!} = \cos\theta$
    - Odd ($n = 2k+1$): $i\sum(-1)^k \frac{\theta^{2k+1}}{(2k+1)!} = i\sin\theta$
3. Therefore $e^{i\theta} = \cos\theta + i\sin\theta$

Deriving Trig Identities from Complex Exponentials

- From $e^{i(a+b)} = e^{ia} \cdot e^{ib}$: expand both sides, equate real and imaginary parts $\rightarrow$ angle addition formulas
- **Double angle**: use $(e^{ix})^2 = e^{2ix}$. Expand $(\cos x + i\sin x)^2 = \cos^2 x - \sin^2 x + 2i\sin x\cos x$. Real part: $\cos(2x) = \cos^2 x - \sin^2 x$

Lectures 22–23: Double Integrals

Key Concepts

- $\iint_D f(x,y)\,dA$: integral of f over region D
- **Order of integration**: $\int\int dy\,dx$ vs $\int\int dx\,dy$ — choose whichever gives simpler limits

Method: Evaluate a Double Integral

1. **Sketch the region**: draw boundary curves, find intersection points
2. **Choose integration order**: pick the order with simpler limits
    - If the integrand is hard to integrate in one variable (e.g., $\cos(x^2)$), try the other order
3. **Set up limits**:
    - For $dy\,dx$: x goes from a to b, y goes from $g_1(x)$ to $g_2(x)$
    - For $dx\,dy$: y goes from c to d, x goes from $h_1(y)$ to $h_2(y)$
4. **Split if needed**: if the boundary changes form, split into sub-regions
5. **Evaluate**: inner integral first (treat outer variable as constant), then outer integral

Method: Interchange Order of Integration

1. Sketch the region from the original limits
2. Re-describe boundaries in the other variable
3. Rewrite the integral with the new limits

Quick Reference: When to Use What

Problem Type	Method
Limit with $0/0$	Factor/cancel, conjugate, or L'Hopital
Limit with 1^∞, 0^0, ∞^0	Rewrite as $e^{g \ln f}$, L'Hopital on exponent
Limit at ∞ with oscillation	Squeeze theorem
Asymptotes	VA: denom $= 0$; HA: $\lim_{x\to\pm\infty}$; OA: $a = \lim f/x$, $b = \lim(f - ax)$
Linearization	$L(x) = f(a) + f'(a)(x - a)$
Implicit dy/dx	Differentiate both sides, chain rule on y-terms, solve for dy/dx
Extreme values (single var)	$f' = 0$ + endpoints, compare f-values
Integration	Substitution, IBP (LIATE), partial fractions
Odd/even symmetry	Odd on $[-a, a] \to 0$; even $\to 2\int_0^a$
Improper integral	Replace ∞ with limit; split at singularities; p-test
Recursive sequence limit	Monotone + bounded $\to$ converges; solve $L = f(L)$
Series convergence	Divergence test $\to$ ratio/root/comparison $\to$ alternating series test
Series sum	Geometric formula, differentiation trick, partial fractions/telescoping
Interval of convergence	Ratio/root test for R; check endpoints separately
$f^{(k)}(0)$ from series	Find c_k in the Taylor series; $f^{(k)}(0) = k! \cdot c_k$
Limit via Taylor	Expand to sufficient order, simplify, evaluate
Anti-derivative via series	Integrate term-by-term; alternating error $\leq$ first omitted term
Gradient	$\nabla f = (f_x, f_y)$
Directional derivative	$D_{\mathbf{u}} f = \nabla f \cdot \hat{\mathbf{u}}$
Critical points (multivariable)	Solve $\nabla f = 0$; classify with $D = f_{xx}f_{yy} - f_{xy}^2$
Absolute max/min on domain	Interior critical pts + boundary edges + corners; compare all f-values
Double integral	Sketch region, choose order, set up limits, evaluate inside-out
Complex arithmetic	Conjugate for division; polar for powers
Complex roots $z^n = c$	$z_k = r^{1/n} e^{i(\theta + 2\pi k)/n}$
Euler's formula	Power series of $e^{i\theta}$, separate even/odd terms